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Flowing with eight supersymmetries in M-theory and F-theory

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Abstract: We consider holographic RG flow solutions with eight supersymmetries and study the geometry transverse to the brane. For both $M2$-branes and for $D3$-branes in F-theory this leads to an eight-manifold with only a four-form flux. In both settings there is a natural four-dimensional hyper-Kähler slice that appears on the Coulomb branch. In the IIB theory this hyper-Kähler manifold encodes the Seiberg-Witten coupling over the Coulomb branch of a U(1) probe theory. We focus primarily upon a new flow solution in M-theory. This solution is first obtained using gauged supergravity and then lifted to eleven dimensions. In this new solution, the brane probes have an Eguchi-Hanson moduli space with the $M2$-branes spread over the non-trivial 2-sphere. It is also shown that the new solution is valid for a class of orbifold theories. We discuss how the hyper-Kähler structure on the slice extends to some form of G-structure in the eight-manifold, and describe how this can be computed.

Keywords: M-Theory, Superstring Vacua, AdS-CFT and dS-CFT Correspondence, Supersymmetry and Duality.

Contents

1. Introduction

There remains a body of important open questions in the study of holographic flows in AdS/CFT, and more generally in the study of string vacua: The problem is to find a geometric characterization of *supersymmetric* backgrounds that involve non-trivial fluxes. There are, of course, well established theorems for varying levels of supersymmetry in purely metrical backgrounds, and these theorems involve hyper-Kähler, Kähler and "Special" geometry. There has also been a lot of work on supersymmetry breaking using fluxes in Calabi-Yau compactifications. These solutions often do not incorporate the back-reaction of the branes, or the flux, or both upon the geometry. When they do consider the back-reaction, the fluxes are typically arranged so as to yield a Ricci-flat background. Thus such solutions rarely provide exact, large N solutions with the AdS space-times that one appears to need for holography.

For AdS backgrounds, there is a very significant body of work on the classification of solutions that involve wrapping N 5-branes (see, for example, [1]–[4]). Such backgrounds seem to be amenable to classifications using G-structures, however they are also relatively simple in that they typically involve only the NS 3-form flux and the dilaton, or they involve the S-dual configuration. The problem of classifying supersymmetric AdS compactifications in the presence of multiple RR-fluxes, or both NS and RR fluxes, remains wide open, particularly for compactifications to four and five dimensions.

Perhaps the most vexatious example of this open problem is the construction of the holographic duals of "Seiberg-Witten" flows. In particular, one would like to find the general holographic dual of the flow in which one gives a mass to a hypermultiplet in the large-N, $\mathcal{N}=4$ Yang-Mills theory, and flows to the Coulomb branch of $\mathcal{N}=2$, large N, Yang-Mills theory in the infra-red. In terms of the $D3$-branes, the Coulomb branch at large-N is represented by a distribution function for the branes. On the Coulomb branch of the $\mathcal{N}=4$ theory, all six scalars can develop vevs, and the distribution function is an arbitrary function of all six variables transverse to the branes. It is this function that sources the familiar harmonic function that underpins these solutions. In an $\mathcal{N}=2$ theory, there are two real scalars in the vector multiplet, and so the Coulomb branch should be represented by an arbitrary function of the two coordinates corresponding to these scalars. Crudely speaking, the complex Higgs invariants, u_n, $n=1,\dots,N$ should be the coefficients of some form of series expansion of the density distribution function of the branes. Thus the supergravity dual of the general flow should involve an arbitrary source function of two variables.

More generally, one would like to know the geometric principle that determines, or classifies, half-maximal supersymmetry[1] in the presence of non-trivial fluxes. Without fluxes the relevant criterion is hyper-Kähler geometry, which is extremely restrictive in low dimensions. Because the presence of eight supersymmetries so strongly constrains the geometry without fluxes, it seems that this amount of supersymmetry is a reasonable starting point to investigate supersymmetric geometry in the presence of fluxes.

We are by no means going to solve these problems here, but we hope that this work will assist in its ultimate solution: One of the obstructions to solving the problem is the paucity of examples. To date, there is only one known multiple-flux, "Seiberg-Witten" flow solution [5], and the corresponding brane distribution function was determined in [6, 7]. Naively this brane distribution looks like the branes are smeared out uniformly on a disk, but in reality this "uniform-disk" distribution is flattened onto a line distribution [6, 7]. This solution of IIB supergravity was found via the methods of gauged $\mathcal{N}=8$ supergravity in five dimensions [8, 9, 10], and this distribution arises because it is, in some sense, the most uniform distribution of branes. The problem is to generalize this and find the solution to IIB supergravity with an arbitrary two-dimensional distribution function. The difficulty is that these solutions involve non-trivial backgrounds for *all* the tensor gauge fields, as well as for the dilaton and axion of the IIB theory. The "simplest" solution of [5] is still far too complicated for easy generalizations.

[1] Here we are taking "maximal" to mean 16 supersymmetries, which is the maximal number in the presence of a brane.

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A useful first step is to try and recast the solution of [5] in a simpler form. One way to accomplish this is to lift the solution to F-theory, and then the dilaton and axion will be encoded in the metric, while the 3-form field strengths are promoted to 4-forms. The geometry transverse to the $D3$-branes is then eight-dimensional, and the most complicated forms are "half-dimensional" on the internal eight-manifold. This seems very appropriate if one is expecting to generalize the notion of hyper-Kähler. Indeed, as we will discuss in section 2, the F-theory lifts of *general* Seiberg-Witten flows cleanly encode the dilaton and axion into a four-dimensional hyper-Kähler slice of the underlying eight-manifold.

A related, and more direct way of trying to generate eight-manifolds associated with compactifications that have eight supersymmetries is to go directly to M-theory and study flows there. Such flows are, of course, related by T-duality to flows in the IIB theory, and the geometries are necessarily very similar (see, for example, [11]). In this paper we will find the M-theory analog of the IIB flow solution of [5]. It is geometrically much simpler: The transverse manifold is eight-dimensional, and there is only a 4-form field strength. There is also a close relationship with the correspondingly maximally supersymmetric Coulomb flow. We also find, once again that there is a four-dimensional hyper-Kähler slice analogous to that of the F-theory lift of the $D3$-brane solution.

To be more precise, we study in detail a two parameter family of flows. One parameter involves a scalar vev that preserves an $\mathrm{SO}(4) \times \mathrm{SO}(4)$ symmetry, and the other involves turning on a fermion mass term that breaks this symmetry down to $(\mathrm{SU}(2) \times \mathrm{U}(1))^2$. A flow with the scalar vev alone preserves 16 supersymmetries, and generates a standard Coulomb branch flow that may be described by the usual harmonic rule with the branes spread out onto a solid 4-ball that is rotated by one of the $\mathrm{SO}(4)$ factors. Turning on the fermion mass reduces the number of supersymmetries to eight.

As in [6, 7] we probe our solution with $M2$-branes, and we find a four-dimensional slice of our solution upon which the potential of the brane probe vanishes identically: It is a slice that corresponds to the Coulomb branch of the theory on $M2$-branes. We also find that the metric on this space of moduli is exactly the Eguchi-Hanson metric, except that there is a conical singularity at the origin. When the fermion mass is non-zero, the branes that were spread on a solid 4-ball on the pure Coulomb branch, are localized on the non-trivial 2-sphere of the Eguchi-Hanson space. We thus obtain a beautifully simple hyper-Kähler section through our solution. We also discuss a more general class of orbifold flows that have eight supersymmetries, and for which the brane-probe moduli space is the Eguchi-Hanson space everywhere, with no singularities.

There are thus several purposes to this paper. First, we find and elucidate a relatively simple flow geometry with eight supersymmetries. There is only one other explicitly known example in the same class, and so we are in this sense doubling the number of examples. We have also found the residual hyper-Kähler geometry in this solution, and in its type IIB bretheren. In section 2 we will describe how such geometry arises naturally in the F-theory lifts of Seiberg-Witten geometries, and how it appears implicitly in the results of [6, 7]. We also discuss how it is related to the hyper-Kähler geometry of the moduli space of $M2$-brane probes.

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In section 3 we start the discussion of our new class of flows. We approach this problem in the same manner as [5, 12, 13]: We first obtain the flows in gauged $\mathcal{N} = 8$ supergravity in four dimensions. In a small aside, we conclude section 3 by noting that our results show that there is a two parameter family of solutions to M-theory in which the parameters trade metric deformations for background fluxes. Indeed, when the parameters are zero, the flow is purely the harmonic metric perturbation corresponding to a Coulomb branch flow. Changing the parameters smoothly turns on background fluxes. Members of the family of flows generically have no supersymmetry, but there is a one parameter family that preserves eight supersymmetries, and another that preserves only four supersymmetries.

Those who do not care for the details of $\mathcal{N} = 8$ supergravity should skip to section 4 where we give the corresponding solution in eleven dimensions. We also show that our new solution is a very close, M-theory analog of the solution of [5]. In particular we examine the geometric symmetries and their actions on the supersymmetries, and identify the two directions within the transverse eight-geometry that represent the two "added" directions when compared to the transverse six-geometry of the $D3$-brane flow given in [5]. We also discuss orbifolds of our solution that preserve all eight supersymmetries.

In section 5 we first look at the pure Coulomb branch flow (the one with 16 supersymmetries) and exhibit its canonical harmonic form. We then use $M2$-brane probes and find the loci upon which they feel "zero force." One of these loci corresponds to the directions in which the branes spread on the Coulomb branch, but when the fermion mass is turned on we show that the metric on the space in the Eguchi-Hanson metric, and that the branes localize on the non-trivial 2-sphere of the Eguchi-Hanson.

Finally, section 6 contains our conclusions as well as some discussion about how we expect to be able to generalize the hyper-Kähler forms out into the full eight-geometry of our solution.

2. Yang-Mills couplings and Hyper-Kähler moduli spaces

It follows from the work of [14] that an $\mathcal{N} = 4$ supersymmetric sigma model must be hyper-Kähler. In particular, given a perturbative

$M2$-brane background that preserves eight supersymmetries (and is thus $\mathcal{N} = 4$ supersymmetric on the world volume), the transverse coordinates of the brane must give rise to a hyper-Kähler sigma-model on the world-volume. There is a possible complication at large-N in that the theory on the $M2$-brane is conformal, strongly coupled, and there is no known explicit description of the field theory. On the other hand one can still use brane probes, and if there is a zero-force domain for the probe, then the corresponding moduli space will necessarily be hyper-Kähler.

For $D3$-branes in IIB supergravity, eight supersymmetries mean $\mathcal{N} = 2$ supersymmetry and thus *a priori* one merely has an ordinary Kähler moduli space. However, if one rewrites this as an F-theory compactification, then the inclusion of the elliptic fiber, T, of F-theory will typically promote the Kähler moduli space to a hyper-Kähler space. This does not follow directly from the IIB picture since there is not enough supersymmetry on the $D3$-brane, and there are no sigma model dynamics in the elliptic fiber. However, one can establish this result using duality: Wrap one of the $D3$-brane directions around a circle, C, and T-dualize. The resulting IIA background can then be thought of as M-theory compactified on a torus with the same complex structure as T, but whose Kähler modulus is the radius of C. In the M-theory, the moduli space of the $M2$-branes is the torus T fibered over the original moduli space of the $D3$-branes. Moreover, since the M-theory solution was obtained as the dual of $D3$-branes, the $M2$ branes will be "smeared" over the circles of the torus.

One can see this very naturally in the $D3$-brane description of $\mathcal{N} = 2^*$ supersymmetric flows from $\mathcal{N} = 4$ Yang-Mills. In such flows the space transverse to the branes is topologically $\mathbb{R}^6$, with the coordinates split into the $2+4$ corresponding to the scalars of the $\mathcal{N} = 2$ vector and hypermultiplets respectively. Let z be a complex coordinate corresponding to the former, so that displacing the branes in the z-direction corresponds to deforming the theory out onto the $\mathcal{N} = 2$ Coulomb branch. A brane probe will therefore experience no force if it approaches the solution along the z-plane through the origin, and the value of the IIB dilaton/axion, $\tau(z)$, is the complex gauge coupling of the U(1) on the probe. Because of the usual properties of $\mathcal{N} = 2$ Yang-Mills, $\tau(z)$ is a holomorphic function.

Let ds_2^2 be the metric felt by the probe on the $\mathcal{N} = 2$ Coulomb branch. The scalar kinetic term for the Yang-Mills bosons provides the canonical metric on the moduli space of the probe:

$$ds_2^2 = \tau_2\,|dz|^2 = e^{-\phi}\,|dz|^2\,, \qquad \text{where } \tau = \tau_1 + i\tau_2 = \frac{\theta_s}{2\pi} + \frac{i}{g_s}\,. \tag{2.1}$$

It is, of course guaranteed within the field theory that $\mathcal{I}m(\tau) > 0$, and indeed this is one of the essential ingredients in obtaining the Seiberg-Witten effective action [15, 16].

Within F-theory, modular invariance on the elliptic fiber fixes the form of the fiber metric, and so the 4-metric of the fibration over (2.1) must be:

$$ds_4^2 = \frac{1}{\tau_2}\,|d\varphi_1 + \tau\,d\varphi_2|^2 + \tau_2\,|dz|^2 = \frac{1}{\tau_2}\,(d\varphi_1 + \tau_1\,d\varphi_2)^2 + \tau_2\,(d\varphi_2^2 + dx^2 + dy^2)\,, \tag{2.2}$$

where $z = x + iy$. This is precisely the kind of metric that was considered in the early investigation of stringy cosmic strings [17, 18, 19]. Remember that $\tau = \tau(z)$ is holomorphic, and thus $\tau_1(x, y)$ and $\tau_2(x, y)$ are harmonic functions that satisfy the Cauchy-Riemann equations. Using this one can easily verify that (2.2) is Ricci flat, with a anti-self-dual curvature tensor. That is, (2.2) is a "half-flat" 4-metric with SU(2) holonomy, and is thus hyper-Kähler. There are two covariantly constant spinors of the same helicity, and the three closed forms that make up the hyper-Kähler structure are:

$$\begin{aligned} J &= \frac{i}{2\,\tau_2}\,(d\varphi_1 + \tau\,d\varphi_2) \wedge (d\varphi_1 + \bar\tau\,d\varphi_2) + \frac{i\,\tau_2}{2}\,dz \wedge d\bar z = d\varphi_1 \wedge d\varphi_2 + \tau_2\,dx \wedge dy \\ \Omega &= (d\varphi_1 + \tau\,d\varphi_2) \wedge dz\,, \qquad \overline{\Omega} = (d\varphi_1 + \bar\tau\,d\varphi_2) \wedge d\bar z\,. \end{aligned} \tag{2.3}$$

The metric (2.2) has a very similar form to the Gibbons-Hawking ALE metrics [20]:

$$ds^2 = V^{-1}\,(d\varphi + A)^2 + V\,(\,d\vec{x} \cdot d\vec{x}\,)\,, \tag{2.4}$$

where

$$V(x) \equiv 2\,m \sum_{j=1}^{M} \frac{1}{|\vec{x} - \vec{x}_j|}\,, \qquad \vec{\nabla} \times \vec{A} = \vec{\nabla} V\,. \tag{2.5}$$

Indeed, one may think of (2.2) as being exactly of this form, but with the point sources in the potential being replaced by line-sources wrapped around the compactified x_3 direction. The harmonic potential and vector field are then $V(x) = \tau_2, \vec{A} = (\tau_1, 0, 0)$, and the relationship between V and $\vec{A}$ reduces to the Cauchy-Riemann equations.

Following the discussion of [6], recall that in terms of the field theory, the probe and the original N $D3$-branes represent the breaking of $\mathrm{SU}(N+1) \to \mathrm{U}(1) \times \mathrm{U}(1)^{N-1}$. On the Coulomb branch, the complex scalar field in the vector multiplet develops the generic vev:

$$\Phi = \mathrm{diag}\Big(z, a_1 - \frac{z}{N}, a_2 - \frac{z}{N}, \ldots, a_N - \frac{z}{N}\Big)\,, \qquad \sum_{j=1}^{N} a_j = 0\,. \tag{2.6}$$

If one computes $\tau(z)$ for the Wilsonian effective action as in [15, 16], $\tau(z)$ is holomorphic, with $\mathcal{I}m(\tau) > 0$, and so the metric (2.2) will be non-singular, except when there are massless BPS states, which appear precisely when the probe brane encounters any of the other "fixed" branes.

In the large-N limit, the effective action of the $\mathcal{N} = 2^*$ gauge theory reduces to its perturbative, one loop form, and so

$$\tau(z) = \frac{i}{g_s} + \frac{\theta_s}{2\,\pi} + \frac{i}{2\,\pi} \sum_{j=1}^{N} \ln\left[\frac{(z - a_j + \frac{z}{N})^2}{(z - a_j + \frac{z}{N})^2 - m^2}\right]. \tag{2.7}$$

However, in AdS/CFT the discrete set of branes at points a_j will be also replaced by a continuum distribution. In particular, for the $D3$ brane flow of [5] it was shown in [6, 7] that:

$$\tau(z) = \frac{i}{g_s}\left(\frac{z^2}{z^2 - a_0^2}\right)^{\frac{1}{2}} + \frac{\theta_s}{2\,\pi}\,, \tag{2.8}$$

corresponding to a linear brane density distribution with density:

$$\rho(x) = \sqrt{a_0^2 - x^2}\,, \tag{2.9}$$

along the $x = \mathcal{R}e(z)$ axis between $-a_0$ and a_0.

Because (2.7) and (2.8) are merely perturbative forms of τ, the imaginary part of τ can vanish at finite values of u, and the metric (2.2) will be singular at such points. For the solution with (2.8), one has $\mathcal{I}m(\tau) = 0$ along the $\mathcal{R}e(z)$ axis between $-a_0$ and a_0, which is precisely where the branes are located. Thus one is really dealing with singular moduli spaces from which the brane sources will need to be excised. Indeed, it has been suggested that such singularities can be resolved in the AdS/CFT correspondence through the enhançon mechanism in which one excises the interior of the region where $\mathcal{I}m(\tau)$ vanishes, and replaces this region by a flat parameter space.

In this paper we will be focusing on $M2$-brane flows, and so we expect that the brane distributions will not be smeared around circles of compactification, but will be localized to the four-dimensional space of moduli. Indeed, if the moduli space is non-trivial then one should expect that it will have the form given by (2.4) and (2.5), and this is indeed what we find.

3. Flows in four-dimensional supergravity

We will describe some holographic flows in the maximally supersymmetric field theory on a stack of $(N+1)$ $M2$-branes. There are eight dimensions transverse to the branes, and so the $\mathcal{R}$-symmetry is SO(8). The theory on a single brane consists of eight bosons, X^I, transforming in the 8_v of SO(8), and eight fermions, λ^a, transforming in the 8_c. This theory has eight supersymmetries, $Q^{a'}$, transforming in the 8_s, and acting through triality. On a stack of $N+1$ such branes there are $8(N+1)$ such bosons and fermions corresponding to the supermultiplet of the transverse locations of the branes. Perturbatively, these would correspond to scalars in the Cartan subalgebra of the underlying $\mathrm{SU}(N+1)$ gauge group, but the superconformal theory only emerges in a strong coupling limit [21]. More precisely, the field theory corresponding to the center of mass motion remains a free theory, but the other degrees of freedom have an infra-red, $\mathcal{N}=8$ superconformal fixed point at which the coupling goes to infinity. This field theory also arises as a Kaluza-Klein reduction of $\mathcal{N}=4$ supersymmetric Yang-Mills theory on a circle. The extra scalars in three-dimensions come from the Wilson line parameter around the circle and from dualizing the three-dimensional photon.

Gauged $\mathcal{N}=8$ supergravity in four dimensions contains 70 scalar fields, and these are holographically dual to operators that have a perturbative form of (traceless) bilinears in the scalars and fermions:

$$\begin{aligned}
\mathcal{O}^{IJ} &= \mathrm{Tr}(X^I\,X^J) - \frac{1}{8}\,\delta^{IJ}\;\mathrm{Tr}(X^K\,X^K)\,, \qquad I,J,\ldots=1,\ldots,8\\
\mathcal{P}^{AB} &= \mathrm{Tr}(\lambda^A\,\lambda^B) - \frac{1}{8}\,\delta^{AB}\;\mathrm{Tr}(\lambda^C\,\lambda^C)\,, \qquad A,B,\ldots=1,\ldots,8\,,
\end{aligned}\tag{3.1}$$

where $\mathcal{O}^{IJ}$ transforms in the $\mathbf{35}_v$ of SO(8), and $\mathcal{P}^{AB}$ transforms in the $\mathbf{35}_c$. Thus, gauged $\mathcal{N}=8$ supergravity in four dimensions can be used to study mass perturbations, and a uniform subsector of the Coulomb branch of the $\mathcal{N}=8$ field theory on the large-N stack of $M2$-branes. The gauged $\mathcal{N}=8$ supergravity in four dimensions thus plays a very analogous role to the gauged $\mathcal{N}=8$ supergravity in five dimensions. There is, however, a significant difference: the Yang-Mills theory has a freely choosable (dimensionless) coupling constant and θ-angle, and these are dual to a pair of scalars in the five-dimensional gauged supergravity theory. The scalar-fermion theory on the $M2$ branes has no free coupling: The perturbative gauge coupling flows to infinity in the fixed point theory. There are thus no supergravity fields dual to a coupling: there are only masses and vevs in the dual of the four-dimensional gauged supergravity.

3.1 A simple family of flows

We wish to consider flows with a pair of independent fermion masses and a pair of independent boson masses. We would like them to be parallel to the two-mass flows in four dimensions in which the fermion masses are $m_1\lambda^3\lambda^3 + m_2\lambda^4\lambda^4$. In the three-dimensional theory we therefore want to consider perturbations involving operators of the form:

$$\begin{aligned}\mathcal{O}_1 &\equiv a_1\,(\mathcal{O}^{11}+\mathcal{O}^{22}+\mathcal{O}^{33}+\mathcal{O}^{44}) + a_2\,(\mathcal{O}^{55}+\mathcal{O}^{66}) + a_3\,(\mathcal{O}^{77}+\mathcal{O}^{88})\,,\\ \mathcal{O}_2 &\equiv b_1\,(\mathcal{P}^{11}+\mathcal{P}^{22}+\mathcal{P}^{33}+\mathcal{P}^{44}) + b_2\,(\mathcal{P}^{55}+\mathcal{P}^{66}) + b_3\,(\mathcal{P}^{77}+\mathcal{P}^{88})\,,\end{aligned} \tag{3.2}$$

where tracelessness requires $a_1 + a_2 + a_3 = 0$ and $b_1 + b_2 + b_3 = 0$.

In the half-maximal supersymmetric flow of [5] in four dimensions the perturbing operators were:

$$\mathcal{O}_b = \sum_{j=1}^{4} \mathrm{Tr}\,(X^j X^j) - 2\sum_{j=5}^{6} \mathrm{Tr}(X^j X^j)\,, \qquad \mathcal{O}_f = \mathrm{Tr}\,(\lambda^3\lambda^3 + \lambda^4\lambda^4)\,, \tag{3.3}$$

and the analogues of these operators on the $M2$-brane are given by (3.2), with $a_2 = a_3 = -a_1$, $b_1 = 0, b_2 = -b_3$.

Since we wish to focus upon flows that involve very particular operators we need to truncate the 70 scalars to a consistent subsector. A simple way to accomplish this is to truncate to the singlet sector of a carefully chosen symmetry. Each of the operators in (3.2) is invariant under $\mathrm{SO}(4)\times\mathrm{SO}(2)\times\mathrm{SO}(2)$, but the SO(4)'s are not the same since the indices in $\mathcal{O}_1$ and $\mathcal{O}_2$ are SO(8) vectors and spinors respectively. The common invariance group is thus $H \equiv \mathrm{SU}(2) \times (\mathrm{U}(1))^3$, and we will ultimately focus on the flows with this invariance group.

3.2 Another approach to the Ansatz

It is useful to consider the other manner in which we might have generalized the results of [5]. Recall that this solution is invariant under $\mathrm{SU}(2)\times\mathrm{U}(1) \subset \mathrm{SO}(4) \subset \mathrm{SO}(6)$. Another natural generalization is to consider the scalar manifold with precisely the same kind of invariance. That is, consider the scalars invarant under $H_0 \equiv \mathrm{SU}(2)\times\mathrm{U}(1)$, where the U(1) lies in one of the SU(2)'s in $(\mathrm{SU}(2))^4 \subset \mathrm{SO}(8)$. Within SO(8), the group H_0 commutes with $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, where this U(1) is the same as the U(1) that appears in H_0. We will drop this U(1) in the commutant henceforth. Within SU(8), H_0 commutes with $\mathrm{SO}(6) \times \mathrm{U}(1)$ and within $E_{7(7)}$ it commutes with $\mathrm{SO}(6,1) \times \mathrm{SL}(2,\mathbb{R})$. Conversely, this latter group commutes with SO(5), and so H_0 is the $\mathrm{SO}(3) \times \mathrm{SO}(2)$ subgroup of SO(5) that commutes with $\mathrm{SO}(6,1)\times\mathrm{SL}(2,\mathbb{R})$ in $E_{7(7)}$. The invariant scalar manifold is therefore:

$$\mathcal{S}_0 \equiv \frac{\mathrm{SO}(6,1)}{\mathrm{SO}(6)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\,. \tag{3.4}$$

There are thus, *a priori*, eight scalars. Six of them may be thought of as being an SO(6) vector, $\vec{v}$. However, the residue of the SO(8) symmetry acts on $\mathcal{S}$ as $\mathrm{SO}(3)\times\mathrm{SO}(3) \subset \mathrm{SO}(6)$, and this may be used to reduce $\vec{v}$ to the form $(v_1, 0, 0, v_4, 0, 0)$, which has two parameters and is manifestly invariant under $\mathrm{SO}(2)\times\mathrm{SO}(2)\subset\mathrm{SO}(3)\times\mathrm{SO}(3)$. This extends H_0 to $H=\mathrm{SU}(2)\times(\mathrm{U}(1))^3$, and indeed this special "gauge" has reduced the apparently more general scalar manifold, $\mathcal{S}$, to the simpler one:

$$\mathcal{S}\equiv\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\times\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \tag{3.5}$$

that is invariant under H and whose scalars are dual to the operators (3.2). Either generalization yields the same end result.

For future reference, it is useful to note that each $\mathrm{SL}(2,\mathbb{R})$ contains two supergravity scalars: one dual to a scalar bilinear in $\mathcal{O}_1$, and the other dual to a scalar bilinear in $\mathcal{O}_2$. In terms of group theory: The non-compact generators in each of the $\mathrm{SL}(2,\mathbb{R})$'s consist of one from the 35_v of $\mathrm{SO}(8)$ and the other from the 35_c. The compact generators come from the 35_s that extends $\mathrm{SO}(8)$ to $\mathrm{SU}(8)$, and so the compact generators are not, *a priori*, symmetries of the theory.

3.3 The flows in gauged supergravity

Following [22, 23], define the action of the $E_{7(7)}$ by:

$$\delta\, z_{IJ}=\Sigma_{IJKL}\, z^{KL}\delta\, z^{IJ}=\Sigma^{IJKL}\, z_{KL}\,, \tag{3.6}$$

where indices are raised and lowered by complex conjugation, and where one has:

$$\Sigma_{IJKL}=\overline{(\Sigma^{IJKL})}=\frac{1}{24}\,\varepsilon_{IJKLPQRS}\,\Sigma^{PQRS}\,.$$

Introduce a complex, skew, self-dual, tensor $X_{ab}=-X_{ba}$, $a,b=1,\dots,4$, by setting:

$$X_{12}=\overline{X}_{34}=z_1\,,\qquad X_{13}=-\overline{X}_{24}=z_2\,,\qquad X_{23}=\overline{X}_{14}=z_3\,. \tag{3.7}$$

The non-compact generators of $\mathrm{SO}(6,1)\times\mathrm{SL}(2,\mathbb{R})$ may then be written as:

$$\Sigma_{IJKL}=24\,(\,z_0\,\delta^{1234}_{[IJKL]}+\bar{z}_0\,\delta^{5678}_{[IJKL]}\,)+24\,X_{ab}\,(\delta^1_{[I}\delta^2_J+\delta^3_{[I}\,\delta^4_J)\,\delta^{a+4}_K\,\delta^{b+4}_{L]}\,, \tag{3.8}$$

where z_0 parametrizes the $\mathrm{SL}(2,\mathbb{R})$, and z_j, $j=1,2,3$ parametrizes the non-compact generators of $\mathrm{SO}(6,1)$. The gauge choice that reduces the scalar manifold to (3.5) is to set $z_2=z_3=0$. The real and imaginary parts of $z_j=x_j+iy_j$, $j=0,\dots,3$ correspond to the 35_v and 35_c and so parametrize the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively.

To get a sense of what these generators represent, one can take $y_j=0$ and contract the Σ's with suitably chosen gammma matrices, and define:

$$S_{AB}\equiv\Sigma_{IJKL}\,(\Gamma^{IJKL})_{AB}\,.$$

One then finds that S is diagonal, and of the form:

$$S=\mathrm{diag}(-\alpha,\,-\alpha,\,-\alpha,\,-\alpha,\,\alpha-2\,\chi,\,\alpha-2\,\chi,\,\alpha+2\,\chi,\,\alpha+2\,\chi)\,, \tag{3.9}$$

which shows how these scalars map onto (3.2).

It is straightforward to exponentiate the actions, (3.6), of these group generators and construct the 56×56 matrices that underlie the $\mathcal{N} = 8$ supergravity in four dimensions. One can then assemble the potential and SU(8) tensors that are central to the structure of the gauged $\mathcal{N} = 8$ theory.

We will use polar coordinates and take:

$$z_0 = \frac{1}{2}\,\alpha\, e^{i\,\phi}\,, \qquad z_1 = \frac{1}{2}\,\chi\, e^{i\,\varphi}\,, \qquad z_2 = z_3 = 0\,. \tag{3.10}$$

If the angles, ϕ and φ, vanish then the supergravity scalars are in the 35_v and correspond to the operators in $\mathcal{O}_1$, and if $\phi = \varphi = \frac{\pi}{2}$, then the supergravity scalars are in the 35_c and correspond to the operators in $\mathcal{O}_2$.

One finds that the scalar kinetic term has the form:

$$(\partial_\mu\, \alpha)^2 + \frac{1}{4}\, \sinh^2(2\,\alpha)\, (\partial_\mu\, \phi)^2 + 2\left[(\partial_\mu\, \chi)^2 + \frac{1}{4}\, \sinh^2(2\,\chi)\, (\partial_\mu\, \varphi)^2\right], \tag{3.11}$$

which determines the canonically normalized scalars to be $\beta_1 = 2\,\chi$ and $\beta_2 = \sqrt{2}\,\alpha$.

Rather surprisingly (given that the compact generators of (3.5) are not a symmetry of the overall theory) one finds that the supergravity potential is completely independent of the angles φ and ϕ, and is given by the simple formula:

$$\mathcal{P} = -\frac{1}{L^2}\,(\cosh(2\,\alpha) + 2\,\cosh(2\,\chi))\,, \tag{3.12}$$

where we have replaced the usual supergravity gauge coupling, g, according to:

$$g = \frac{1}{\sqrt{2}\,L}\,. \tag{3.13}$$

With these conventions, the maximally symmetric critical point at $\alpha = \chi = 0$ gives rise to an AdS_4 vacuum of radius L.

The SU(8) tensor, A_1^{ij}, that appears in the gravitino variation is real and has constant eigenvectors with eigenvalues:

$$\begin{aligned}
\mathcal{W}_1 &= \cosh(\alpha)\, \cosh^2(\chi) + e^{-i\,\phi}\, \sinh(\alpha)\, \sinh^2(\chi)\,, \\
\mathcal{W}_2 &= \cosh(\alpha)\, \cosh^2(\chi) + e^{-i\,(2\,\varphi - \phi)}\, \sinh(\alpha)\, \sinh^2(\chi)\,, \\
\mathcal{W}_3 &= \cosh(\alpha)\, \cosh^2(\chi) + e^{i\,(2\,\varphi + \phi)}\, \sinh(\alpha)\, \sinh^2(\chi)\,,
\end{aligned} \tag{3.14}$$

with multiplicities $4, 2$ and 2 respectively. The constancy of the eigenvectors makes these eigenvalues good candidates for superpotentials [24].

One can easily verify that all of these eigenvalues are related to the potential, $\mathcal{P}$, via:

$$\mathcal{P} = \frac{1}{L^2}\left|\frac{\partial \mathcal{W}}{\partial \alpha}\right|^2 + \frac{1}{2\,L^2}\left|\frac{\partial \mathcal{W}}{\partial \chi}\right|^2 - \frac{3}{L^2}\,|\mathcal{W}|^2\,. \tag{3.15}$$

The absence of derivatives with respect to ϕ and φ here means that the $\mathcal{W}_j$ will only be superpotentials if the angles are fixed in a manner consistent with supersymmetry.

As is conventional, we will consider a flow metric of the form:

$$ds^2_{1,3} = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2\,, \tag{3.16}$$

where $\eta_{\mu\nu}$ is the flat, Poincaré invariant metric of the $M2$-brane. The supersymmetric flow equations are then:

$$\frac{d\alpha}{dr} = -\frac{1}{L}\frac{\partial\mathcal{W}}{\partial\alpha}\,, \qquad \frac{d\chi}{dr} = -\frac{1}{2L}\frac{\partial\mathcal{W}}{\partial\chi}\,, \qquad \frac{dA}{dr} = \frac{1}{L}\mathcal{W}\,. \tag{3.17}$$

There are several natural choices of superpotential. The simplest is to take $\phi = \varphi = 0$ and then all the $\mathcal{W}_j$ are equal. The corresponding flows involve only the operators in $\mathcal{O}_1$, preserve $\mathcal{N} = 8$ supersymmetry, and represent pure "Coulomb branch" flows from the UV fixed point theory.

The flow that we will focus on here comes from taking:

$$\phi = 0\,, \qquad \varphi = \frac{\pi}{2}\,. \tag{3.18}$$

From (3.9) we then see that z_0 preserves $\mathrm{SO}(4)\times\mathrm{SO}(4)$, and then z_1 breaks this down to $(\mathrm{SU}(2)\times\mathrm{U}(1))^2$. For this family of scalars, the superpotential is given by:

$$\begin{aligned}\mathcal{W} = \mathcal{W}_1 &= \cosh(\alpha)\,\cosh^2(\chi) + \sinh(\alpha)\,\sinh^2(\chi)\\ &= \frac{1}{2}\left(\rho^{-1} + \rho\,\cosh(2\,\chi)\right)\,,\end{aligned} \tag{3.19}$$

where $\rho \equiv e^\alpha$. We could equally well have used $\mathcal{W} = \mathcal{W}_2 = \mathcal{W}_3$, which is related to $\mathcal{W}_1$ by $\alpha \to -\alpha$. The important point is that with the choice (3.18) there are two sets of four distinct eigenvalues of A_1^{ij}, and hence the flows are going to be $\mathcal{N} = 4$ supersymmetric; that is, the flows will have half-maximal supersymmetry. Indeed, the four unbroken supersymmetries are singlets under one $\mathrm{SU}(2)\times\mathrm{U}(1)$ factor, and transform as a $\mathbf{2}_{\pm 1}$ of the other $\mathrm{SU}(2)\times\mathrm{U}(1)$.

In terms of the dual theory on the brane, one can see from (3.9), (3.10) and (3.18) that these flows involve the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ with $a_2 = a_3 = -a_1$; $b_1 = 0, b_2 = -b_3$. As noted after (3.3), this is indeed the precise analogue of the flow in [5].

As in [5], one can solve the flow equations completely, and one finds:

$$\alpha = \frac{1}{2}\log\left[e^{-2\,\chi} + \gamma\,\sinh(2\,\chi)\right]\,, \tag{3.20}$$

with

$$e^A = \frac{k\,\rho}{\sinh(2\,\chi)}\,, \tag{3.21}$$

where k and γ are constants of integration. The solution (3.20) is considerably simpler than the analogue in [5], and indeed there are some interesting special cases with $\alpha = \pm\chi$ and $\rho^2 = \cosh(2\chi)$. The solution (3.20) has different asymptotics depending upon whether γ is positive, negative or zero. Since the superpotential has a manifest symmetry under $\chi \to -\chi$, we focus on $\chi > 0$: If γ is positive then $\alpha \sim \chi + \frac{1}{2}\log(\gamma/2)$ for large (positive) χ. If γ is negative then χ limits to a finite value, χ_0, as α goes to $-\infty$. If $\gamma = 0$ then we get the

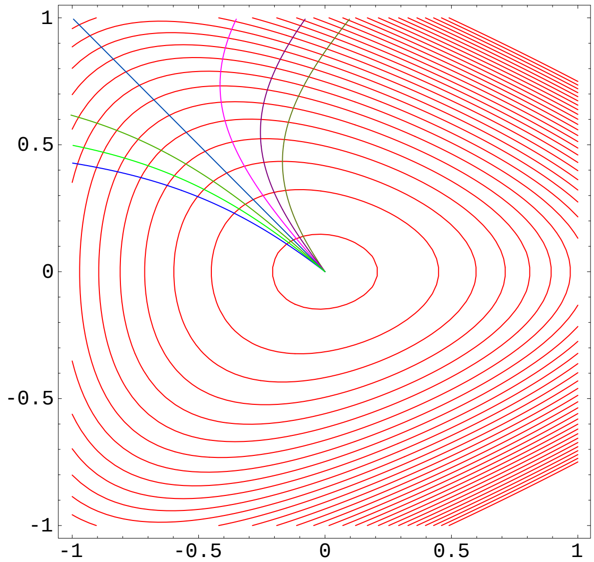

Figure 1: Contours of the superpotential showing some of the steepest descent flows. The horizontal and vertical axis are α and χ, respectively. The ridge-line flow has $\gamma = 0$ and the three flows to the left and right have $\gamma < 0$ and $\gamma > 0$, respectively. To the right of the ridge the flows asymptote to $\alpha \sim \chi$, on the ridge one has $\alpha = -\chi$, and to the left of the ridge one has $\alpha \to -\infty$ as χ limits to a finite value.

interesting ridge-line flow with $\alpha \equiv -\chi$ all along the flow. Some of these flows are shown in figure 1. In all three cases, one has $e^A \to 0$, and so the metric, (3.16), becomes singular.

These results precisely parallel those of [5], and suggest that the "preferred flow," analogous to the purely massive Yang-Mills flow in four dimensions, is the one with $\gamma = 0$. Following the parallel with the $D3$-brane flow, the solution with $\gamma < 0$ correspond to flows in which the vevs of the operator $\mathcal{O}_1$ dominates over the mass of the fermions. It thus approaches the Coulomb branch in the infra-red. The solutions with $\gamma > 0$ are presumably unphysical as they are for $D3$-branes. We will see later, the structure of the M-theory uplift is very similar to that of IIB uplift of the $D3$-brane flow [5]. We will also see that our claim about the $\gamma = 0$ flow is supported by the $M2$-brane probes.

3.4 An Aside: Continuous families of flow solutions

The supergravity potential (3.12) exhibits a very curious feature that leads to a surprising conclusion in eleven dimensional geometry. In its original form, the gauged $\mathcal{N} = 8$ supergravity theory has a local $SO(8) \times SU(8)$ symmetry. The scalar manifold is $E_{7(7)}/\,SU(8)$, and, as is usual, we have fixed the SU(8) symmetry by chosing symmetric gauge in which all the scalars are represented by exponentials of the seventy non-compact generators of $E_{7(7)}$. There is still an SU(8) *conjugation* action on this scalar manifold, but it is *only* the SO(8) subgroup that is symmetry of the theory: Complex SU(8) transformations map the four-dimensional scalars into the pseudo-scalars and vice-versa. In (3.6) the supergravity scalars and pseudo-scalars are represented by the real and imaginary parts of the self-dual, complex forms, Σ_{IJKL}. At the linearized level, the scalars are internal metric perturbations while the pseudo-scalars are internal components of the 3-form, A_{MNP}.

It follows from our discussion above that the U(1) rotations defined by the phase angles ϕ and φ in (3.10) are complex SU(8) rotations, and the real and imaginary parts of the z_j are scalars and pseudo-scalars respectively. Turning on only the real part of the z_j represents a pure metric deformation, while turning on the imaginary part represents turning on a tensor gauge field background, which will then affect the metric at second order through the back-reaction. Rotations of ϕ and φ thus trade metric deformations for tensor gauge fields, and vice-versa. The statement that such SU(8) rotations are not symmetries of the theory is a simply the statement that such a rotation of the configurations will not generically preserve the lagrangian, and thus will not be consistent with the equations of motion.

However, the potential (3.12) does *not* depend upon ϕ and φ, and indeed both the potential and the kinetic term (3.11) are invariant under translations in ϕ and φ. Thus, given a solution at with some fixed value of ϕ and φ, we can trivially generate another by rotating the value of ϕ and φ. In particular, the pure Coulomb branch solution with $\phi = \varphi = 0$ can be rotated into a two parameter, continuous family of solutions with non-trivial background fluxes. Therefore, there is a family of solutions that interpolates between the pure metric flow with $\phi = \varphi = 0$, and the flow with (3.18) considered above.

The flow with $\phi = \varphi = 0$ has $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3$, and if we use the equations of motion (3.17) then this flow has sixteen supersymmetries. This solution (with $\phi = \varphi = 0$ has $\mathcal{W} = \mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3$) actually solves the second order system of equations coming form (3.11) and (3.12) for *any* fixed values of ϕ and φ. However, the resulting flow will not be supersymmetric for general values of ϕ and φ. Partial supersymmetry will be present in special cases: For example, there will be eight supersymmetries if $\phi = 0$, and four supersymmetries if $\phi = \pm 2\varphi$.

Thus the unexpected symmetry of the potential (3.12) enables us to generate continuous families of solutions of M-theory in which metric deformations are traded for fluxes, and supersymmetry is partially, or totally broken within these families, with extra supersymmetries appearing on special flows. A particularly simple, but illustrative example of this is studied in detail in [25].

4. The flow solutions in M-theory

4.1 The new solution

From the large body of work on consistent truncation, and most particularly from [26, 27, 28], we know that the solution given in the last section can be lifted to eleven dimensions. First, the metric for the eleven-dimensional solution can be obtained directly from the scalar field configuration by employing the metric formula in [26]. For the flows with (3.18) we find the following frames:

$$\begin{aligned}
e^1 &= \Omega\, e^A\, dt\,, \qquad e^2 = \Omega\, e^A\, dx\,, \qquad e^3 = \Omega\, e^A\, dy\,, \qquad e^4 = \Omega\, dr\,, \\
e^5 &= 2\, L\, c^{-\frac{1}{2}}\, \Omega\, d\theta\,, \qquad e^6 = L\, \rho\, \Omega\, X_1^{-\frac{1}{2}} \cos\theta\, \sigma_1\,, \qquad e^7 = L\, \rho\, \Omega\, X_1^{-\frac{1}{2}} \cos\theta\, \sigma_2\,, \\
e^8 &= L\, \rho\, \Omega\, (c\, X_2)^{-\frac{1}{2}} \cos\theta\, \sigma_3\,, \qquad e^9 = L\, \Omega\, X_2^{-\frac{1}{2}} \sin\theta\, \tau_1\,, \\
e^{10} &= L\, \Omega\, X_2^{-\frac{1}{2}} \sin\theta\, \tau_2\,, \qquad e^{11} = L\, \Omega\, (c\, X_1)^{-\frac{1}{2}} \sin\theta\, \tau_3\,,
\end{aligned} \tag{4.1}$$

where $c \equiv \cosh(2\chi)$ and $\rho = e^{\alpha}$. The functions X_1, X_2 and Ω are defined by:

$$
\begin{aligned}
X_1 &= (\cos^2\theta + \rho^2\,\cosh(2\chi)\,\sin^2\theta)\,, \qquad X_2 = (\cosh(2\chi)\,\cos^2\theta + \rho^2\,\sin^2\theta)\,, \\
\Omega &= (\rho^{-2}\,\cosh(2\chi)\,X_1\,X_2)^{\frac{1}{6}}\,.
\end{aligned} \tag{4.2}
$$

The σ_j and τ_j are independent sets of left-invariant one-forms on SU(2), and satisfy $d\sigma_1 = \sigma_2 \wedge \sigma_3$; $d\tau_1 = \tau_2 \wedge \tau_3$, plus cyclic permutations. This metric has a manifest symmetry of $(\mathrm{SU}(2)\times \mathrm{U}(1))^2$, and there is also an interchange symmetry:

$$
\theta \to \frac{\pi}{2} - \theta\,, \qquad \alpha \to -\alpha\,. \tag{4.3}
$$

The equations of motion (in the conventions of [29, 30]) are:

$$
\begin{aligned}
R_{MN} + R\,g_{MN} &= \frac{1}{3}\,F^{(4)}_{MPQR}\,F^{(4)\,PQR}_{N}\,, \\
\nabla_M F^{MNPQ} &= -\frac{1}{576}\,\epsilon^{NPQR_1R_2R_3R_4S_1S_2S_3S_4}\,F_{R_1R_2R_3R_4}\,F_{S_1S_2S_3S_4}\,.
\end{aligned} \tag{4.4}
$$

Based upon previous experience [5, 31, 32], we conjecture the following form for the 3-form potential:

$$
\begin{aligned}
A^{(3)} &= \widetilde{W}\,e^{3A}\,dt\wedge dx\wedge dy + a_1(\chi,\theta)\,\sin\theta\,\cos\theta\,d\theta\wedge\sigma_3\wedge\tau_3 + \\
&\quad + a_2(\chi,\theta)\,\sin^2\theta\,\cos^2\theta\,\sigma_1\wedge\sigma_2\wedge\tau_3 + a_3(\chi,\theta)\,\sin^2\theta\,\cos^2\theta\,\sigma_3\wedge\tau_1\wedge\tau_2\,.
\end{aligned} \tag{4.5}
$$

The quantity, $\widetilde{W}$ is sometimes called the geometric superpotential, and its form can be inferred by techniques similar to those of [32, 11]. We indeed find that:

$$
\widetilde{W} = \frac{1}{2}\,\rho^{-1}\,X_1 = \frac{1}{2}\,\rho^{-1}\,(\cos^2\theta + \rho^2\,\cosh(2\chi)\,\sin^2\theta)\,. \tag{4.6}
$$

We find that the Ricci tensor of the metric obeys the identities:

$$
R_{11} = -R_{22} = -R_{33}\,, \qquad R_{66} = R_{77}\,, \qquad R_{99} = R_{10\,10}\,, \tag{4.7}
$$

and

$$
R_{66} + R_{88} = R_{11}\,, \qquad R_{99} + R_{11\,11} = R_{11}\,. \tag{4.8}
$$

The identities (4.7) are a trivial consequence of the symmetries of the metric, while the identities (4.8) are consistent with the special Anstatz, (4.5), for the 3-form.

To determine the functions $a_j(\chi,\theta)$, we proceed much as in [5]. One computes the field strength, F, and one finds that it has the form:

$$
\begin{aligned}
F &= h_1\,e^1\wedge e^2\wedge e^3\wedge e^4 + h_2\,e^1\wedge e^2\wedge e^3\wedge e^5 + b_1\,e^4\wedge e^5\wedge e^8\wedge e^{11} + \\
&\quad + b_2\,e^4\wedge e^6\wedge e^7\wedge e^{11} + b_3\,e^4\wedge e^8\wedge e^9\wedge e^{10} + b_4\,e^5\wedge e^6\wedge e^7\wedge e^{11} + \\
&\quad + b_5\,e^5\wedge e^8\wedge e^9\wedge e^{10} + b_6\,e^6\wedge e^7\wedge e^9\wedge e^{10}\,,
\end{aligned} \tag{4.9}
$$

where the h's and b's are given in terms of the functions $\widetilde{W}$ and a_j. One then computes the energy-momentum tensor in terms of the h's and b's, and from this one can immediately test, and verify that (4.6) yields the correct geometric superpotential. One then looks at differences of components of the energy momentum tensor, and compares them with the corresponding differences between components the Ricci tensor, using (4.6) where necessary. The result is typically an analytic expression for the difference of squares of two of the b_k. One can then factorize both sides and find the individual b_k. The result is then integrated to get the a_j. We find:

$$\begin{aligned} a_1(\chi,\theta) &= L^3 \tanh(2\,\chi)\,, \qquad a_2(\chi,\theta) = \frac{1}{2}\,L^3\,\rho^2\,X_1^{-1}\sinh(2\,\chi)\,, \\ a_3(\chi,\theta) &= \frac{1}{2}\,L^3\,X_2^{-1}\sinh(2\,\chi)\,. \end{aligned} \tag{4.10}$$

A tedious, but straightforward calculation shows that the 3-form defined by (4.5) and (4.10) satisfies the generalized Maxwell equations in (4.4). We thus have a complete "lift" of the supersymmetric flows of section 2.

The results for both the metric and the potential are very similar in form to the result for the corresponding IIB flow given in [5]. For the IIB flow the metric on the 5-sphere had the form:

$$ds_5^2 = \frac{a^2}{2}\frac{(cX_1X_2)^{1/4}}{\rho^3}\left(c^{-1}d\theta^2 + \rho^6\cos^2\theta\left(\frac{\sigma_1^2}{cX_2} + \frac{\sigma_2^2+\sigma_3^2}{X_1}\right) + \sin^2\theta\,\frac{d\phi^2}{X_2}\right), \tag{4.11}$$

while the complex 2-form potential was:

$$A_{(2)} = e^{i\phi}\left(a_1(r,\theta)\,\cos\theta\,d\theta\wedge\sigma_1 + a_2(r,\theta)\,\sin\theta\cos^2\theta\,\sigma_2\wedge\sigma_3 + a_3(r,\theta)\,\sin\theta\cos^2\theta\,\sigma_1\wedge d\phi\right), \tag{4.12}$$

with

$$\begin{aligned} a_1(r,\theta) &= -i\,L^2\,\tanh(2\chi)\,, \qquad a_2(r,\theta) = i\,L^2\,\rho^6\,X_1^{-1}\,\sinh(2\chi)\,, \\ a_3(r,\theta) &= L^2\,X_2^{-1}\,\sinh(2\chi)\,. \end{aligned} \tag{4.13}$$

The functions, X_j, are very similar to those considered here, just with powers of ρ changed consistently throughout. As one can see, the forms are remarkably similar, and indeed, to collapse the eight-geometry to the six-geometry one roughly drops the τ_3 everywhere, and replaces τ_1 and τ_2 by $d\phi$. If one writes:

$$\begin{aligned} \tau_1 &\equiv \cos(\phi_3)\,d\phi_1 + \sin(\phi_3)\,\sin(\phi_1)\,d\phi_2\,, \\ \tau_2 &\equiv \sin(\phi_3)\,d\phi_1 - \cos(\phi_3)\,\sin(\phi_1)\,d\phi_2\,, \\ \tau_3 &\equiv \cos(\phi_1)\,d\phi_2 + d\phi_3\,, \end{aligned} \tag{4.14}$$

then the foregoing suggests that one should identify ϕ with some combination of ϕ_1 and ϕ_2, while the other combination, which we will denote ψ should be combined with ϕ_3 to get the compactifying circle for the M-theory. The remaining combination of ψ and ϕ_3 will then become the extra dimension that takes the $M2$-brane to the $D3$-brane. One can, in fact track the detailed identification through the gauged supergravities in four and five dimensions. We will not pursue this in detail here because the T-dual of the IIB solution of [5] is not exactly the same as our M-theory solution: The former involves a smeared distribution of $D3$-branes, while the latter involves a $M2$-branes that are localized in $\mathbb{R}^8$.

4.2 Supersymmetries and orbifolds

It is very instructive to examine which of the geometric symmetries act as $\mathcal{R}$-symmetries. Denote the symmetries of the metric and 3-form by $\mathrm{SU}(2)_\sigma \times \mathrm{U}(1)_\sigma$ and $\mathrm{SU}(2)_\tau \times \mathrm{U}(1)_\tau$, where the subscripts denote which of the left-invariant one-forms are involved. While we have not computed the supersymmetries in eleven dimensions explicitly, one can use the results of section 3 and either the linearized solution, or the metric uplift formula of [26] to relate the geometric symmetries on S^7 to the symmetries acting in the four-dimensional $\mathcal{N} = 8$ theory. Doing this, we find that the unbroken supersymmetries are singlets under $\mathrm{SU}(2)_\tau \times \mathrm{U}(1)_\sigma$, and that they transform as $\mathbf{2}_{\pm 1}$ under $\mathrm{SU}(2)_\sigma \times \mathrm{U}(1)_\tau$. One can check that these transformation properties are consistent with these supersymmetry parameters becoming the Weyl components of the $\mathcal{N} = 2$ supersymmetry in the IIB theory after a T-duality in one of the τ-directions.

Finally, we note that because the supersymmetry parameters are inert under $\mathrm{SU}(2)_\tau$, the $\mathcal{N} = 4$ supersymmetry will be *preserved* under any orbifold by a discrete subgroup of $\mathrm{SU}(2)_\tau$. Indeed, such an orbifold construction can be used *ab initio* to reduce the supersymmetry as in [33, 34, 35] to the half-maximal amount. The flows considered here may thus be thought of the flows in the untwisted sector of the resulting $\mathcal{N} = 4$ supergravity in four dimensions. This closely parallels the corresponding discussion for gauged supergravity in five dimensions (see, for example, [36, 37]).

The bottom line is that the flows considered here may be thought of as flows within the untwisted sectors of an ADE family of orbifold theories obtained by modding out the corresponding discrete group of $\mathrm{SU}(2)_\tau$.

5. Brane distributions and probes

5.1 The harmonic distribution

To understand the brane distribution in our solution, we first examine the simplest situation in which the fermion bilinears are set to zero. A flow that only involves the operator $\mathcal{O}_1$ of (3.2), is a pure Coulomb branch flow, and it must preserve maximal supersymmetry. In supergravity such a flow lies purely in the scalar sector, and must correspond to a standard harmonic distribution of branes. Indeed, if one makes the further restriction of setting $\chi = 0$ in (3.10) then one must obtain an $\mathrm{SO}(4) \times \mathrm{SO}(4)$ invariant distribution. This solution is easily mapped out, and correponds to a uniform distribution of branes on a solid 4-ball.

To see this one simply sets $\chi = 0$ in the equations of motion (3.17). The eleven-dimensional metric becomes:

$$\begin{aligned} ds_{11}^2 = {} & H^{-2/3}\,(-dt^2 + dx^2 + dy^2) + \\ & + H^{1/3}\,\frac{4\,L^2}{\sinh\alpha}\left(\frac{X_0}{4\,\rho\,L^2}\,dr^2 + \frac{X_0}{\rho}\,d\theta^2 + \frac{1}{4}\,\rho\,\cos^2\theta\sum_{j=1}^{3}\sigma_j^2 + \frac{1}{4}\,\rho^{-1}\,\sin^2\theta\sum_{j=1}^{3}\tau_j^2\right), \end{aligned} \tag{5.1}$$

where

$$H \equiv e^{-3A}\,\rho\,X_0^{-1} = \rho\,\sinh^3\alpha\,X_0^{-1}\,, \qquad X_0 \equiv X_1|_{\chi=0} = (\cos^2\theta + \rho^2\,\sin^2\theta)\,. \tag{5.2}$$

By making a change of variables:

$$u = \frac{e^{\alpha}}{\sqrt{(e^{2\,\alpha}-1)}}\,\cos\theta\,, \qquad v = \frac{1}{\sqrt{(e^{2\,\alpha}-1)}}\,\sin\theta\,, \tag{5.3}$$

the metric reduces to the standard harmonic form:

$$\begin{aligned} ds_{11}^2 &= H^{-2/3}(-dt^2+dx^2+dy^2)+ \\ &\quad +8\,H^{1/3}\,L^2\left(du^2+dv^2+\frac{1}{4}\,u^2\sum_{j=1}^{3}\sigma_j^2+\frac{1}{4}\,v^2\sum_{j=1}^{3}\tau_j^2\right), \end{aligned} \tag{5.4}$$

where the metric in the second set of parentheses is the flat metric on $\mathbb{R}^8$. One can then check that in these new coordinates, the harmonic function, $H(u,v)$, of (5.2) is given by:

$$H(u,v) = \text{const.}\int_{z^2<1}\frac{d^4z}{(u^2+(\vec{v}-\vec{z})^2)^3}\,, \tag{5.5}$$

where $\vec{v}$ and $\vec{z}$ are vectors in $\mathbb{R}^4$. This means that the $M2$ branes are spread out into a solid 4-ball, defined by $v^2<1$, with a constant density of branes throughout the ball.

5.2 The general distribution: Brane probes

One can attempt to find the general distribution of branes by naively trying to force the general eleven-metric into a the "harmonic form" (5.1). That is, one now writes the general metric in the form:

$$ds_{11}^2 = H^{-2/3}\,ds_{2,1}^2 + H^{1/3}\,ds_8^2\,, \tag{5.6}$$

where

$$H \equiv e^{-3A}\,\Omega^{-3}\,, \tag{5.7}$$

and

$$\begin{aligned} ds_8^2 \equiv \Omega^3\,e^A\,L^2\bigg[&\frac{1}{L^2}\,dr^2+\frac{4}{c}\,d\theta^2+\frac{\rho^2}{X_1}\,\cos^2\theta\,(\sigma_1^2+\sigma_2^2)+\frac{\rho^2}{c\,X_2}\,\cos^2\theta\,\sigma_3^2+ \\ &+\frac{1}{X_2}\,\sin^2\theta\,(\tau_1^2+\tau_2^2)+\frac{1}{c\,X_1}\,\sin^2\theta\,\tau_3^2\bigg]\,. \end{aligned} \tag{5.8}$$

These expressions are only a formal parallel, and the function H is certainly not harmonic, but H should encode the brane distribution. The way to do this correctly is, of course, to use brane probes.

The action for the membrane probe has the following pieces:

$$S = I^{DBI} + I^{WZ} \equiv \mu_2\int d^3\sigma\left[-\sqrt{-det(\tilde{g})}+\frac{1}{3}\,\tilde{A}^{(3)}\right], \tag{5.9}$$

where $\tilde{g}$ and $\tilde{A}^{(3)}$ denote the pull-back of the metric and the 3-form onto the membrane. The normalization of the $A^{(3)}$-term in (5.9) is twice the usual normalization since this is the normalization that we have used in the eleven-dimensional equations of motion.

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Following the usual approach, we take the probe to be parallel to the source membranes, and assume that it is traveling at a small velocity transverse to its world-volume. Expanding to second order in velocities, one obtains a kinetic energy and a potential term:

$$I^{\rm probe} \equiv \int t(T-V) = \frac{1}{2} m_{M2} \int dt \, \Big(G_{m\,n}\, v^m\, v^n - e^{3A}\, (\Omega^3 - 2\,\widetilde{W}) \Big)\,, \tag{5.10}$$

where we have replaced the μ_2 and the integral over the spatial volume of the probe by the mass m_{M2}. In this expression, the indices $m, n = 1, \ldots, 8$ run over the directions transverse to the brane, and the metric G_{mn} is precisely that of (5.8):

$$ds_8^2 \equiv G_{mn}\, dx^m\, dx^n = \Omega\, e^A \Big(g_{mn}\, dx^m\, dx^n \Big)\,, \tag{5.11}$$

where the metric g_{mn} is obtained from the last eight frames in (4.1).

The potential seen by the probe membrane is thus:

$$V = m_{M2}\, e^{3A}\, (\,\Omega^3 - 2\,\widetilde{W}\,) = \frac{1}{2}\, m_{M2}\, \frac{k^3\, \rho^2}{\sinh^3 2\,\chi}\, \sqrt{X_1}\, \Big(\sqrt{c\, X_2} - \sqrt{X_1} \Big)\,. \tag{5.12}$$

This is very similar to the result found in [6, 7].

The force-free regions, or moduli spaces of the brane probe, are given by the loci where the potential vanishes. From (5.12) one sees that there are two such regions:

$$\textbf{I}: \quad \sqrt{c\,X_2} = \sqrt{X_1} \quad \Leftrightarrow \quad \cos\theta = 0\,.$$

$$\textbf{II}: \quad \rho = 0\,.$$

Recall that the harmonic branes were spread in the v-direction, which corresponds to $\cos\theta = 0$, and so this is going to be the physically interesting direction. We will therefore briefly consider locus II, and study locus I in more detail.

5.3 The special loci

Locus II only exists for the flows with $\gamma < 0$ in (3.20). Let χ_0 be the value of χ at which ρ vanishes, and define $c_0 = \cosh(2\chi_0)$, $s_0 = \sinh(2\chi_0)$. As one $\rho \to 0$, one has, from (3.17) and (3.19):

$$dr \sim 2\, L\, d\rho\,. \tag{5.13}$$

The metric, ds_8^2, becomes, to leading order in ρ:

$$ds_8^2 \sim \frac{4\,k\,L^2}{s_0} \Big[c_0 \Big(du^2 + \frac{1}{4}\, u^2 \Big(\sigma_1^2 + \sigma_2^2 + \frac{1}{c_0^2}\, \sigma_3^2 \Big) \Big) + dw^2 + \frac{1}{4}\, w^2\, (\tau_1^2 + \tau_2^2 + \tau_3^2) \Big]\,, \tag{5.14}$$

where $u \equiv \rho\cos\theta$ and $w \equiv \sin\theta$. Note that to leading order, w does not involve ρ, and thus w remains finite in the range $|w| \le 1$ as $\rho \to 0$. In this same limit, the brane-distribution function, (5.7), is, to leading order:

$$H \sim \frac{s^3}{c}\, \frac{1}{u^2}\,. \tag{5.15}$$

This is once again consistent with a uniform distribution of M_2-branes spread in a four-dimensional ball. At $\rho = 0$ the branes thus see the completely flat moduli space, parametrized by w.

This result is not too surprising since the $\gamma < 0$ solutions correspond to a solutions in which the Coulomb branch vevs of the scalars are dominating over the fermion mass terms, and so in the infra-red, the probe brane should see the moduli space of the original Coulomb branch.

The physically most interesting flows are those with $\gamma = 0$. These are the ones that should describe the purely massive flows with no Coulomb branch component to the flow. For such flows locus II is singular, and there only remains locus I. The moduli space is particularly interesting.

Setting $\theta = \pi/2$ in (5.8) yields the 4-metric:

$$ds_4^2 = \frac{c}{s^3}\, d\chi^2 + \frac{c}{4\, s}\,(\tau_1^2 + \tau_2^2) + \frac{1}{4\, c\, s}\tau_3^2\,, \tag{5.16}$$

where $c = \cosh(2\chi)$, $s = \sinh(2\chi)$. There are several remarkable features to note immediately. First, ρ has cancelled out completely, and so this limiting metric is independent of the choice of γ. Secondly, as $\chi \to \infty$ the 2-sphere defined by τ_1 and τ_2 limits a sphere of radius 1/2. Indeed, define a new radial variable, $\mu = \sqrt{c/s}$ and one finds:

$$ds_4^2 = \frac{d\mu^2}{(1 - \frac{1}{\mu^4})} + \frac{1}{4}\,\mu^2\,(\tau_1^2 + \tau_2^2) + \frac{1}{4}\,\mu^2\left(1 - \frac{1}{\mu^4}\right)\tau_3^2\,, \tag{5.17}$$

which is almost exactly the Eguchi-Hanson metric. It is thus the Ricci-flat, hyper-Kähler metric on the blown-up A_1 singularity. The blown-up 2-sphere, defined by τ_1 and τ_2, has radius 1/2.

It is not exactly the Eguchi-Hanson metric because the periodicity of the angular coordinate in τ_3 is not correct. If one uses (4.14) then the angle ϕ_3 has period $4\,\pi$, but for (5.17) to be regular as $\mu \to 1$, ϕ_3 must have a period of $2\,\pi$. Thus, at large μ, the surfaces of constant μ in the Eguchi-Hanson metric must look like $S^3/\mathbb{Z}_2$ and not like S^3. Our flow solution started with S^7 without any discrete identifications, and so the 4-slice represented by (5.16) does not have the $\mathbb{Z}_2$ identifications needed to get Eguchi-Hanson exactly.

On the other hand, we noted at the end of section 4 that our flow solution and its supersymmetries are invariant under $\mathrm{SU}(2)_\tau$, and so may be modded out by any discrete subgroup $\mathrm{SU}(2)_\tau$. Therefore we can trivially modify our solution to accommodate the asymptotics of any ALE space, and indeed if we go to the $\mathbb{Z}_2$ orbifold then we get the $M2$-brane moduli space to be Eguchi-Hanson on the nose. It would be interesting to try to generalize our solution to get an arbitrary ALE space as the moduli space of the $M2$-branes.

Thus the $M2$-branes see, at least locally, a classic hyper-Kähler moduli space, and as $\chi \to \infty$, or $\mu \to 1$, the branes are spread out over a finite-sized 2-sphere. Indeed, for the $\gamma = 0$ flow, the brane-distribution function, (5.7), exhibits the proper asymptotics. First observe that the coordinates χ and μ are singular as $\chi \to \infty$. Indeed, $ds_4^2 \sim 4e^{-4\,\chi}\, d\chi^2 + \ldots$, which means that a good radial coordinate is $R \equiv e^{-2\chi}$. Then, for $\theta = \frac{\pi}{2}$, and $\gamma = 0$, the brane-distribution function (5.7) becomes:

$$H = \frac{\sinh^3(2\chi)}{\rho^4 \cosh(2\chi)} \sim \frac{1}{4} e^{8\chi} \sim \frac{1}{4}\frac{1}{R^4}\,, \tag{5.18}$$

which is precisely the correct asymptotics for a uniform distribution of $M2$-branes spread over a 2-sphere at $R = 0$. Moreover, for $\theta = \frac{\pi}{2}$ the 3-form potential becomes:

$$A^{(3)} \sim H^{-1}\, dt \wedge dx \wedge dy\,, \tag{5.19}$$

which further supports our interpretation.

It is worth recalling the parallel result for the $D3$-brane flow of [6, 7]. For $D3$-branes the corresponding Coulomb branch involved spreading the $D3$-branes uniformly over a two-dimensional disk. Turning on a fermion mass involved turning on background 2-form gauge fields, and a non-trivial dilaton and axion. The IIB background could thus be thought of as a dielectric mix of additional 5-branes and 7-branes. Using $D3$-brane probes on locus II involved approaching the solution in the two-dimensional direction ($\theta = \pi/2$) where the disk of branes had spread on the pure Coulomb flow. In the more general background, with non-zero fermion masses, the 2-form fields vanished on locus II, but the dilaton and axion remained non-trivial, and could be used to track the distribution of $D3$-branes. (In a sense, they had dissolved in the dielectric 7-branes that sourced the dilaton and axion.) In particular, the $D3$-brane probe of the "purely massive" ($\gamma = 0$) flow was studied extensively in [6, 7]. It was shown that the "Coulomb disk" had flattened to a line distribution of branes in which the brane density was no longer constant, but seemed to "remember" its disk-like origins:

$$\rho(y) = \sqrt{a_0^2 - y^2}\,. \tag{5.20}$$

For the $M2$-brane flow we have found that the four-ball of the Coulomb branch has collapsed, and then been blown up into a 2-sphere with an apparently uniform $M2$-brane distribution.

6. Conclusions

Our motivation in doing this work was to try to get a broader geometric understanding of supersymmetric solutions in the presence of RR-fluxes. We ultimately hope to find a geometric characterization of general supersymmetric holographic flows, and go beyond those that we can access using lower-dimensional gauged supergravity.

In this paper we focused on flow solutions with eight supersymmetries in both F-theory and M-theory. These solutions involve a transverse eight-manifold with half-dimensional forms (i.e. 4-forms) only. We believe that such configurations, along with such a high level of supersymmetry should be more amenable to classification than, say, their IIB cousins with a transverse six-manifold and $1-$, $3-$ and $5-$form fluxes. Within M-theory, if there is a moduli space for the brane probe then a hyper-Kähler structure is guaranteed by the eight supersymmetries. Within IIB supergravity, eight supersymmetries means $\mathcal{N} = 2$

supersymmetry, which only implies a Kähler moduli space. However, we showed that lifting the IIB flows to F-theory extends this to a four-dimensional hyper-Kähler moduli space whose singularities coincide with the enhançon.

Our explicit M-theory flow was constructed to precisely parallel the $\mathcal{N} = 2$ supersymmetric flow in the IIB theory. Given that we started with a topologically trivial transverse space, with no orbifold identifications, we had anticipated finding a Coulomb moduli-space that was topologically $\mathbb{R}^4$. Instead, we found that it was Eguchi-Hanson *without* the proper modding out by $\mathbb{Z}_2$, but with the branes smoothly distributed over the non-trivial S^2. In addition, we noted that we could mod-out our solution so as to preserve the supersymmetry and yet make this Coulomb moduli space have the large radius asymptotics appropriate to any ALE space: S^3/Γ where Γ is any discrete subgroup of SU(2). In particular we can mod out the entire solution by $\mathbb{Z}_2$ so that it still has eight supersymmetries and has exactly the Eguchi-Hanson moduli space.

While the example presented here is far simpler than that of [5], we are still not able to fully characterize the flow geometries with eight supersymmetries. However, the results presented here provide what appears to be a very fruitful line of attack that is very much in the same spirit as using G-structures (see, for example, [38]). First, there is the presence of the four-dimensional hyper-Kähler moduli space on the residual Coulomb branch. While this is very constraining, we would obviously like to understand how this geometry is extended and generalized in directions transverse to the moduli space. It turns out that the eleven-dimensional supersymmetries will tell us precisely how to achieve this. To be more precise, this solution has eight supersymmetries contained in four supersymmetry parameters, $\epsilon_{(i)}$. The group transformation property of these spinors under $\mathrm{SU}(2)_\tau \times \mathrm{U}(1)_\tau$ is precisely consistent with these spinors being made from tensor products that involve the covariantly constant spinors on the Eguchi-Hanson space. Indeed, this is also required for consistency with the supersymmetry transformations on the probe brane. Now consider the bilinears:

$$J^{(ij)}_{\mu\nu} \equiv \bar{\epsilon}_{(i)} \gamma_\mu \gamma_\nu \epsilon_{(j)} \,. \tag{6.1}$$

Because the $\epsilon_{(i)}$ satisfy $\delta\psi_\mu = 0$, the $J^{(ij)}$ must be 2-forms that satisfy first-order differential equations. In the direction of the Eguchi-Hanson space these spinors must be covariantly constant, and these bilinears therefore yield the hyper-Kähler forms. Thus by finding the Killing spinors of the complete solution we will be able to explicitly see how the hyper-Kähler structure is extended away from the Eguchi-Hanson slice. We are currently pursuing this idea, and computing the explicit supersymmetries.

There is one other observation that suggests a special geometry transverse to the brane. In section 3 we made a choice (3.19) of superpotential. We could equally well have made the opposite choice: $\mathcal{W} = \mathcal{W}_2 = \mathcal{W}_3$, which amounts to sending $\alpha \to -\alpha$ in our solution. In section 4 we noted in (4.3) that this was a symmetry of the full solution when coupled with the rotation $\theta \to \frac{\pi}{2} - \theta$. This means that a flow that uses the other choice of superpotential will generate another Eguchi-Hanson slice at $\theta = 0$, which is exactly transverse to the Eguchi-Hanson slice described above. So a relatively trivial change in the equations of motion creates another hyper-Kähler slice transverse to the original one. There will then

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be a directly parallel story with the spinors, and the extension of the hyper-Kähler forms into the whole eight-manifold. By combining all the 2-forms generated in this manner, we hope to be able to pin down the geometry very precisely.

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